

# Regular Interval Exchange Transformations over a Quadratic Field

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**Abstract.** We describe a generalization of a result of Boshernitzan and Carroll: an extension of Lagrange's Theorem on continued fraction expansion of quadratic irrationals to interval exchange transformations. In order to do this, we use a two-sided version of the Rauzy induction. In particular, we show that starting from an interval exchange transformation whose lengths are defined over a quadratic field and applying the two-sided Rauzy induction, one can obtain only a finite number of new transformations up to homothety.

**Keywords:** Symbolic Dynamics, Interval Exchange Transformations, Rauzy Induction, Continued Fractions.

## 1 Introduction

It is a truth universally acknowledged that the simple continued fraction expansion of a quadratic irrational must be eventually periodic (result known as Lagrange's Theorem).

Continued fractions are related to different combinatorial tools, such as Stern-Brocot trees, mechanical words, rotations, etc. (see [7] and [8]). An interesting representation of the continued fraction development is given by inducing the first return map of a 2-interval exchange transformation, the ratio of whose lengths is a quadratic irrational, on the larger exchanged semi-interval.

Interval exchange transformations were introduced by Oseledec [11] following an earlier idea of Arnol'd [1]. These transformations form a generalization of rotations of the circle (the two notions coincide when there are exactly 2 intervals). Rauzy introduced in [9] a transformation, now called Rauzy induction (or Rauzy-Veech induction), which operates on interval exchange transformations. It actually transforms an interval exchange transformation into another, operating on a smaller semi-interval. Its iteration can be viewed, as mentioned, as a generalization of the continued fraction development (since we work with  $n$ -interval exchange transformations with  $n \geq 2$ ). The induction consists in taking the first return map of the transformation with respect to a particular subsemi-interval of the original semi-interval. A two-sided version of Rauzy induction is studied in [2], along with a characterization of the intervals reachable by the iteration of this two-sided induction, the so called admissible intervals.

Interval exchange transformations defined over quadratic fields have been studied by Boshernitzan and Carroll ([4] and [5]). Under this hypothesis, they

showed that, using iteratively the first return map on one of the semi-intervals exchanged by the transformation, one obtains only a finite number of different new transformations up to rescaling, extending the classical Lagrange's theorem that quadratic irrationals have a periodic continued fraction expansion.

In this paper we generalize this result, enlarging the family of transformations obtained using induction on every admissible semi-interval. This contains the results of [5] because every semi-interval exchanged by a transformation is admissible, while for  $n > 2$  there are admissible semi-intervals that we can not obtain using the induction only on the exchanged ones.

The paper is organised as follows.

In Section 2, we recall some notions concerning interval exchange transformations, minimality and regularity. We also introduce an equivalence relation on the set of interval exchange transformations. We finally recall the result of Keane [10] which proves that regularity is a sufficient condition for minimality of such a transformation (Theorem 1).

In Section 3 we recall the Rauzy induction and the generalization to its two-sided version. We also recall the definition of admissibility and how this notion is related to Rauzy induction (Theorems 4 and 5). We conclude the section introducing the equivalence graph of a regular interval exchange transformation.

The final part of this paper, Section 4, is devoted to the proof of our main result (Theorem 6), i.e. the finiteness of the number of equivalence classes for a regular interval exchange transformation defined over a quadratic field.

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## 2 Interval Exchange Transformations

Let us recall the definition of an interval exchange transformation (see [6] or [2] for a more detailed presentation).

A *semi-interval* is a nonempty subset of the real line of the form  $[\ell, r[ = \{z \in \mathbb{R} \mid \ell \leq z < r\}$ . Thus it is a left-closed and right-open interval.

Let  $A = \{a_1, a_2, \dots, a_s\}$  be a finite ordered alphabet with  $a_1 < a_2 < \dots < a_s$  and  $(I_a)_{a \in A}$  an ordered partition of  $[\ell, r[$  in semi-intervals. Set  $\lambda_i$  the length of  $I_{a_i}$ . Let  $\pi \in \mathcal{S}_s$  be a permutation on  $A$ .

Define  $\gamma_i = \sum_{a_j < a_i} \lambda_j$  and  $\delta_{\pi(i)} = \sum_{\pi(a_j) < \pi(a_i)} \lambda_j$ . Set  $\alpha_a = \delta_a - \gamma_a$ . The *interval exchange transformation* relative to  $(I_a)_{a \in A}$  is the map  $T : [\ell, r[ \rightarrow [\ell, r[$  defined by

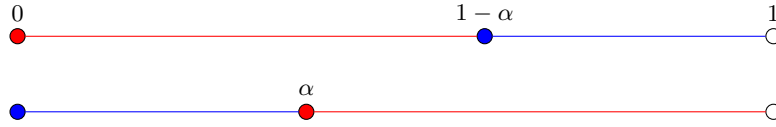
$$T(z) = z + \alpha_a \quad \text{if } z \in I_a.$$

Observe that the restriction of  $T$  to  $I_a$  is a translation onto  $J_a = T(I_a)$ , that  $\gamma_i$  is the left boundary of  $I_{a_i}$  and that  $\delta_j$  is the left boundary of  $J_{a_j}$ .

Note that the family  $(J_a)_{a \in A}$  is also a partition of  $[\ell, r[$ . In particular, the transformation  $T$  defines a bijection from  $[\ell, r[$  onto itself.

An interval exchange transformation relative to  $(I_a)_{a \in A}$  is also called a  $s$ -interval exchange transformation. The values  $(\alpha_a)_{a \in A}$  are called the *translation values* of the transformation  $T$ . We will also denote  $T = T_{\pi, \lambda}$ , where  $\lambda = (\lambda_i)_{a_i \in A}$  is the ordered sequence of lengths of the semi-intervals.

*Example 1.* Let  $T = T_{\pi, \lambda}$  be the interval exchange transformation corresponding to  $A = \{a, b\}$ ,  $a < b$ ,  $\pi = (12)$ , i.e. such that  $\pi(b) < \pi(a)$  and  $\lambda = (1 - \alpha, \alpha)$  with  $\alpha = \frac{3 - \sqrt{5}}{2}$ . Thus the two semi-intervals exchanged by  $T$  are  $I_a = [0, 1 - \alpha[$  and  $I_b = [1 - \alpha, 1[$ . The transformation  $T$ , representend in Figure 1, is the rotation of angle  $\alpha$  on the semi-interval  $[0, 1[$  defined by  $T(z) = z + \alpha \bmod 1$ .



**Fig. 1.** Rotation of angle  $\alpha$  on the semi-interval  $[0, 1[$ .

Note that the transformation  $T_{\pi, \lambda}$ , does not depend on the relative position, (the choice of the left point  $\ell$ ).

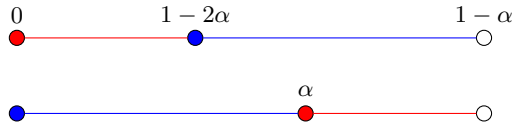
It is easy to verify that the family of  $s$ -interval exchange transformations is closed by taking inverses.

## 2.1 Equivalent Interval Exchange Transformations

Two  $s$ -interval exchange transformation  $T = T_{\pi, \lambda}$  and  $S = T_{\sigma, \mu}$  are said to be *equivalent* either if  $\sigma = \pi$  and  $\mu = c\lambda$  for some  $c > 0$  or if  $\sigma = \tau \circ \pi$  and  $\mu = c\tilde{\lambda}$ , where  $\tau : i \mapsto (s - i + 1)$  is the permutation that reverses the names of the semi-intervals and  $\tilde{\lambda} = (\lambda_s, \lambda_{s-1}, \dots, \lambda_1)$ .

We denote by  $[T_{\pi, \lambda}]$  the equivalence class of  $T_{\pi, \lambda}$ .

*Example 2.* Let  $T_{\pi, \mu}$  be the interval exchange transformation defined by  $\pi = (12)$  and  $\mu = (1 - 2\alpha, \alpha)$ , with  $\alpha = \frac{3 - \sqrt{5}}{2}$  (see Figure 2). The transformation  $T_{\pi, \mu}$  is equivalent to the transformation  $T_{\pi, \lambda}$  of Example 1. Indeed  $\alpha^2 = 3\alpha - 1$  and one can easily show that  $\mu = (1 - \alpha)\tilde{\lambda}$ .



**Fig. 2.** Transformation  $T_{(12), (1-2\alpha, \alpha)}$ .

## 2.2 Regular Interval Exchange Transformations

The *orbit* of a point  $z \in [\ell, r[$  is the set  $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ . The transformation  $T$  is said to be *minimal* if for any  $z \in [\ell, r[$ ,  $\mathcal{O}(z)$  is dense in  $[\ell, r[$ .

The points  $0 = \gamma_1, \gamma_2, \dots, \gamma_s$  form the set of *separation points* of  $T$ , denoted  $\text{Sep}(T)$ . Note that the transformation  $T$  has at most  $s - 1$  *singularities* (points at which it is not continuous), which are among the nonzero separation points  $\gamma_2, \dots, \gamma_s$ .

An interval exchange transformation  $T_{\pi, \lambda}$  is called *regular* if the orbits of the nonzero separation points  $\gamma_2, \dots, \gamma_s$  are infinite and disjoint. Note that the orbit of 0 cannot be disjoint from the others since one has  $T(\gamma_i) = 0$  for some  $i$  with  $2 \leq i \leq s$ .

A regular interval exchange transformation is also said to satisfy the *idoc* (infinite disjoint orbit condition). It is also said to have the Keane property or to be without *connection* (see [3]).

Note that since  $\delta_{\pi(2)} = T(\gamma_2), \dots, \delta_{\pi(s)} = T(\gamma_s)$ ,  $T$  is regular if and only if the orbits of  $\delta_{\pi(2)}, \dots, \delta_{\pi(s)}$  are infinite and disjoint.

As an example, the 2-interval exchange transformation of Example 1 is regular, as every rotation of irrational angle. The following result is due to Keane [10].

**Theorem 1 (Keane).** *A regular interval exchange transformation is minimal.*

The converse is not true. Indeed, consider the rotation of angle  $\alpha$  with  $\alpha$  irrational, as a 3-interval exchange transformation with  $\lambda = (1 - 2\alpha, \alpha, \alpha)$  and  $\pi = (123)$ . The transformation is minimal, as is any rotation of an irrational angle, but it is not regular since  $\gamma_2 = 1 - 2\alpha$ ,  $\gamma_3 = 1 - \alpha$  and thus  $\gamma_3 = T(\gamma_2)$ .

## 3 Rauzy Induction

We recall in this section the transformation called Rauzy induction, defined in [9], which operates on regular interval transformations, and some results concerning this transformation (Theorems 2 and 3). We also recall a two-sided version of this transformation studied in [2] and some of the results relative to it (Theorems 4 and 5).

### 3.1 Right Rauzy Induction

Let  $T = T_{\pi, \lambda}$  be an interval exchange transformation relative to  $(I_a)_{a \in A}$ .

For  $\ell < t < r$ , the semi-interval  $[\ell, t[$  is *right admissible* for  $T$  if there is a  $k \in \mathbb{Z}$  such that  $t = T^k(\gamma_a)$  for some  $a \in A$  and

- (i) if  $k > 0$ , then  $t < T^h(\gamma_a)$  for all  $h$  such that  $0 < h < k$ ,
- (ii) if  $k \leq 0$ , then  $t < T^h(\gamma_a)$  for all  $h$  such that  $k < h \leq 0$ .

We also say that  $t$  itself is right admissible. Note that all semi-intervals  $[\ell, \gamma_a[$  with  $\ell < \gamma_a$  are right admissible. Similarly all semi-intervals  $[\ell, \delta_a[$  with  $\ell < \delta_a$  are right admissible.

*Example 3.* Let  $T$  be the interval exchange transformation of Example 1. The semi-interval  $[0, t[$  for  $t = 1 - \alpha$  or  $t = 1 - 2\alpha$  is right admissible since  $1 - \alpha = \gamma_2$  and  $1 - 2\alpha = T^{-1}(\gamma_2) < \gamma_2$ . On the contrary, for  $t = 2 - 3\alpha$ , it is not right admissible because  $t = T^{-2}(\gamma_2)$  but  $\gamma_2 < t$  contradicting (ii).

Assume now that  $T$  is minimal. Let  $I \subset [\ell, r[$  be a semi-interval. Since  $T$  is minimal, for each  $z \in [\ell, r[$  there exists an integer  $n > 0$  such that  $T^n(z) \in I$ .

The transformation induced by  $T$  on  $I$  is the transformation  $S : I \rightarrow I$  defined for  $z \in I$  by  $S(z) = T^n(z)$  with  $n = \min\{n > 0 \mid T^n(z) \in I\}$ . The semi-interval  $I$  is called the *domain* of  $S$ , denoted  $D(S)$ .

*Example 4.* Let  $T$  be the transformation of Example 1. Let  $I = [0, 1 - \alpha[$ . The transformation induced by  $T$  on  $I$  is

$$S(z) = \begin{cases} T(z) & \text{if } 0 \leq z < 1 - 2\alpha \\ T^2(z) & \text{otherwise.} \end{cases}$$

The following result is Theorem 14 in [9].

**Theorem 2 (Rauzy).** *Let  $T$  be a regular  $s$ -interval exchange transformation and let  $I$  be a right admissible interval for  $T$ . The transformation induced by  $T$  on  $I$  is a regular  $s$ -interval exchange transformation.*

Note that the transformation induced by an  $s$ -interval exchange transformation on  $[\ell, r[$  on any semi-interval included in  $[\ell, r[$  is always an interval exchange transformation on at most  $s + 2$  intervals (see [6], Chapter 5 p. 128).

*Example 5.* Consider again the transformation of Example 1. The transformation induced by  $T$  on the semi-interval  $I = [0, 1 - \alpha[$  is the 2-interval exchange transformation represented in Figure 2.

Let  $T = T_{\pi, \lambda}$  be a regular  $s$ -interval exchange transformation on  $[\ell, r[$ . Set

$$Z(T) = [\ell, \max\{\gamma_s, \delta_{\pi(s)}\}].$$

Note that  $Z(T)$  is the largest semi-interval which is right-admissible for  $T$ . We denote by  $\psi(T)$  the transformation induced by  $T$  on  $Z(T)$ .

The following result is Theorem 23 in [9].

**Theorem 3 (Rauzy).** *Let  $T$  be a regular interval exchange transformation. A semi-interval  $I$  is right admissible for  $T$  if and only if there exists an integer  $n \geq 0$  such that  $I = Z(\psi^n(T))$ . In this case, the transformation induced by  $T$  on  $I$  is  $\psi^{n+1}(T)$ .*

The map  $T \mapsto \psi(T)$  is called the *right Rauzy induction*.

*Example 6.* Consider again the transformation  $T$  of Example 1. Since  $Z(T) = [0, 1 - \alpha[$ , the transformation  $\psi(T)$  is the one represented in Figure 2.

### 3.2 Left Rauzy Induction

The symmetrical notion of *left Rauzy induction* is defined similarly. Define

$$Y(T) = [\min\{\gamma_2, \delta_{\pi(2)}\}, r[.$$

We denote by  $\varphi(T)$  the transformation induced by  $T$  on  $Y(T)$ . The map  $T \mapsto \varphi(T)$  is called the *left Rauzy induction*.

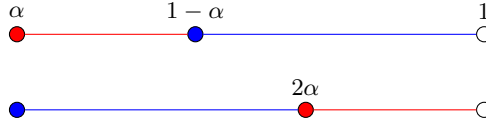
The notion of left admissible interval is symmetrical to that of right admissible. For  $\ell < t < r$ , the semi-interval  $[t, r[$  is *left admissible* for  $T$  if there is a  $k \in \mathbb{Z}$  such that  $t = T^k(\gamma_a)$  for some  $a \in A$  and

- (i) if  $k > 0$ , then  $T^h(\gamma_a) < t$  for all  $h$  such that  $0 < h < k$ ,
- (ii) if  $k \leq 0$ , then  $T^h(\gamma_a) < t$  for all  $h$  such that  $k < h \leq 0$ .

We also say that  $t$  itself is left admissible.

The symmetrical statement of Theorem 3 also holds for left admissible intervals. Note that, similar to the right admissibility, we have  $[\gamma_a, r[$  and  $[\delta_a, r[$  left admissible for every  $a \in A$ .

*Example 7.* Let  $T$  be the transformation of Example 1. One has  $Y(T) = [\alpha, 1[$ . The transformation  $\varphi(T) = T_{(12), (1-2\alpha, 1-\alpha)}$  is represented in Figure 3.



**Fig. 3.** Transformation  $T_{(12), (1-2\alpha, 1-\alpha)}$  induced by  $T$  on  $[\alpha, 1[$ .

Note that for a 2-interval exchange transformation  $T$ , one has  $\psi(T) = \varphi(T)$ , whereas in general the two transformations are different.

### 3.3 Two-sided Induction

In this section, we generalize the left and right Rauzy inductions to a two-sided induction (see [2] for a detailed presentation).

Let  $T = T_{\pi, \lambda}$  be an  $s$ -interval exchange transformation on  $[\ell, r[$  relative to  $(I_a)_{a \in A}$ . For a semi-interval  $I = [u, v[ \subset [\ell, r[$ , we define the following functions on  $[\ell, r[$

$$\rho_{I, T}^+(z) = \min\{n > 0 \mid T^n(z) \in ]u, v[ \}, \quad \rho_{I, T}^-(z) = \min\{n \geq 0 \mid T^{-n}(z) \in ]u, v[ \}.$$

We define the set of *neighbours* of  $z$  with respect to  $I$  and  $T$  as

$$N_{I, T}(z) = \{T^k(z) \mid -\rho_{I, T}^-(z) \leq k < \rho_{I, T}^+(z)\}.$$

The set of *division points* of  $I$  with respect to  $T$  is the finite set

$$\text{Div}(I, T) = \bigcup_{i=1}^s N_{I, T}(\gamma_i).$$

For  $\ell \leq u < v \leq r$ , we say that the semi-interval  $I = [u, v[$  is *admissible* for  $T$  if  $u, v \in \text{Div}(I, T) \cup \{r\}$ .

Note that a semi-interval  $[\ell, v[$  is right admissible if and only if it is admissible and that a semi-interval  $[u, r[$  is left admissible if and only if it is admissible. Note also that  $[\ell, r[$  is admissible.

Note also that for a regular interval exchange transformation relative to a partition  $(I_a)_{a \in A}$ , each of the semi-intervals  $I_a$  (or  $J_a$ ) is admissible although only the first one is right admissible (and the last one is left admissible).

Recall that  $\text{Sep}(T)$  denotes the set of separation points of  $T$ , i.e. the points  $\gamma_1 = 0, \gamma_2, \dots, \gamma_s$  (which are the left boundaries of the semi-intervals  $I_{a_1}, I_{a_2}, \dots, I_{a_s}$ ). The following generalization of Theorem 2 is proved in [2].

**Theorem 4.** *Let  $T$  be a regular  $s$ -interval exchange transformation on  $[\ell, r[$ . For any admissible semi-interval  $I = [u, v[$ , the transformation  $S$  induced by  $T$  on  $I$  is a regular  $s$ -interval exchange transformation with separation points  $\text{Sep}(S) = \text{Div}(I, T) \cap I$ .*

We have already noted that for any  $s$ -interval exchange transformation on  $[\ell, r[$  and any semi-interval  $I$  of  $[\ell, r[$ , the transformation  $S$  induced by  $T$  on  $I$  is an interval exchange transformation on at most  $s+2$ -intervals. Actually, it follows from the proof of Lemma 2, page 128 in [6] that, if  $T$  is regular and  $S$  is an  $s$ -interval exchange transformation with separation points  $\text{Sep}(S) = \text{Div}(I, T) \cap I$ , then  $I$  is admissible. Thus the converse of Theorem 4 is also true.

The following generalization of Theorem 3 is proved in [2].

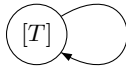
**Theorem 5.** *Let  $T$  be a regular  $s$ -interval exchange transformation on  $[\ell, r[$ . A semi-interval  $I$  is admissible for  $T$  if and only if there exists a sequence  $\chi \in \{\varphi, \psi\}^*$  such that  $I$  is the domain of  $\chi(T)$ . In this case, the transformation induced by  $T$  on  $I$  is  $\chi(T)$ .*

### 3.4 Equivalence Graph

For an interval exchange transformation  $T$  we consider the directed graph  $G(T)$ , called the *equivalence graph* of  $T$ , defined as follows. The vertices are the equivalence classes of transformations obtained starting from  $T$  and applying all possible  $\chi \in \{\psi, \varphi\}^*$ . There is an edge starting from a vertex  $[T]$  to a vertex  $[S]$  if and only if  $S = \theta(T)$  for two transformations  $T \in [T]$  and  $S \in [S]$  and a  $\theta \in \{\psi, \varphi\}$ .

*Example 8.* Let  $T = T_{(12), (1-\alpha, \alpha)}$  be the regular 2-interval exchange transformation of Example 1. Applying the right and the left Rauzy induction on  $T$  we obtain  $\psi(T) = \varphi(T) = T_{(12), (1-2\alpha, \alpha)}$  (see Examples 5). These two transformations are equivalent (see Example 2). Therefore the equivalence graph of  $T$ , represented

in Figure 4, contains only one vertex. Note that the ratio of the two lengths of the semi-intervals exchanged by  $T$  is  $\frac{1-\alpha}{\alpha} = \frac{1+\sqrt{5}}{2} = \phi = 1 + \frac{1}{1+\dots} = 1 + \frac{1}{\phi}$ , i.e. the golden ratio.



**Fig. 4.** Equivalence graph of the transformation  $T = T_{(12), (1-\alpha, \alpha)}$ .

Note that, in general, the equivalence graph can be infinite. We will give in the next section a sufficient condition for the equivalence graph to be finite.

## 4 Interval Exchange Transformations Over a Quadratic Field

An interval exchange transformation is said to be defined over a set  $Q \subset \mathbb{R}$  if the lengths of all exchanged intervals belong to  $Q$ .

The following is proved in [5]. Let  $T$  be a minimal interval exchange transformation on semi-intervals defined over a quadratic number field. Let  $(T_n)_{n \geq 0}$  be a sequence of interval exchange transformation such that  $T_0 = T$  and  $T_{n+1}$  is the transformation induced by  $T_n$  on one of its exchanged semi-intervals  $I_n$ . Then, up to rescaling all intervals  $I_n$  to the same length, the sequence  $(T_n)$  contains finitely many distinct transformations.

In this section we generalize this results and prove that, under the above hypothesis on the lengths of the semi-intervals and up to rescaling, there are finitely many transformations obtained by the two-sided Rauzy induction.

**Theorem 6.** *Let  $T$  be a regular interval exchange transformation defined over a quadratic field. The family of all induced transformation of  $T$  over an admissible semi-interval contains finitely many distinct transformations up to equivalence.*

Note that the previous theorem implies the result of [5]. Indeed every semi-interval exchanged by a transformation is admissible, while for  $n > 2$  there are admissible semi-intervals that we can not obtain using the induction only on the exchanged ones.

The proof of the Theorem 6 is based on the fact that for each minimal interval exchange transformation defined over a quadratic field, a certain measure of the arithmetic complexity of the admissible semi-intervals is bounded.

### 4.1 Complexities

Let  $T$  be an interval exchange transformation on a semi-interval  $[\ell, r[$  defined over a quadratic field  $\mathbb{Q}[\sqrt{d}]$ , where  $d$  is a square free integer  $\geq 2$ . Without



loss of generality, one may assume, by replacing  $T$  by an equivalent interval exchange transformation if necessary, that  $T$  is defined over the ring  $\mathbb{Z}[\sqrt{d}] = \{m + n\sqrt{d} \mid m, n \in \mathbb{Z}\}$  and that all  $\gamma_i$  and  $\alpha_i$  lie in  $\mathbb{Z}[\sqrt{d}]$  (replacing  $[\ell, r[$  if necessary by its equivalent translate with  $\gamma_0 = \ell \in \mathbb{Z}[\sqrt{d}]$ ).

For  $z = m + n\sqrt{d}$  let define  $\Psi(z) = \max(|m|, |n|)$ .

The following Proposition is proved in [5, Proposition 2.2].

**Proposition 1.** *For every  $z \in \mathbb{Z}[\sqrt{d}] \setminus \{0\}$ , one has  $|z|\Psi(z) > \frac{1}{2\sqrt{d}}$ .*

Let  $\mathcal{A}([\ell, r[)$  be the algebra of subsets  $S \subset [\ell, r[$  which are finite unions  $S = \bigcup_j I_j$  of semi-intervals defined over  $\mathbb{Z}[\sqrt{d}]$ , i.e.  $I_j = [\ell_j, r_j[$  for some  $\ell_j, r_j \in \mathbb{Z}[\sqrt{d}]$ . Note that the algebra  $\mathcal{A}([\ell, r[)$  is closed under taking finite unions, intersections and passing to complements in  $[\ell, r[$ .

We define the *complexity*  $\Psi(S)$  and the *reduced complexity*  $\Pi(S)$  of a subset  $S \in \mathcal{A}([\ell, r[)$  as

$$\Psi(S) = \max\{\Psi(z) \mid z \in \partial(S)\} \quad \text{and} \quad \Pi(S) = |S|\Psi(S),$$

where  $\partial(S)$  is the boundary of  $S$  and  $|S|$  stands for the Lebesgue measure of  $S$ .

A key tool to prove Theorem 6 is the following Theorem proved in [5, Theorem 3.1].

**Theorem 7 (Boshernitzan).** *Let  $T$  be a minimal interval exchange transformation on an interval  $[\ell, r[$  defined over a quadratic number field. Assume that  $(Y_n)_{n \geq 1}$  is a sequence of semi-intervals of  $[\ell, r[$  such that the set  $\{\Pi(Y_n) \mid n \geq 1\}$  is bounded. Then the sequence  $S_n$  of interval exchange transformations obtained by inducing  $T$  on  $Y_n$  contains finitely many distinct equivalence classes of interval exchange transformations.*

The following Proposition is proved in [5, Proposition 2.1]. It shows that the complexity of a subset  $S$  and of its image  $T(S)$  differs at most by a constant that depends only on  $T$ .

**Proposition 2.** *There exists a constant  $u = u(T)$  such that for every  $S \in \mathcal{A}([\ell, r[)$  and  $z \in [\ell, r[$  one has*

$$|\Psi(T(S)) - \Psi(S)| \leq u \quad \text{and} \quad \Psi(T(z) - z) \leq u.$$

Moreover, one has  $\Psi(\gamma) \leq u$  and  $\Psi(T(\gamma)) \leq u$  for every separation point  $\gamma$ .

Clearly, by Proposition 2, one also has  $|\Psi(T^{-1}(S)) - \Psi(S)| \leq u$  for every  $S \in \mathcal{A}([\ell, r[)$  and  $\Psi(T^{-1}(z) - z) \leq u$  for every  $z \in [\ell, r[$ .

Combining Proposition 1 and Proposition 2 we easily obtain the following (see also [5, Corollary 2.3]).

**Corollary 1.** *There exists a constant  $c > 0$  such that for every  $n \in \mathbb{Z}$  and  $z \in [\ell, r[$  one has either  $T^n(z) = z$  or  $|T^n(z) - z| > \frac{c}{n}$ .*

The following Proposition, proved in [5, Proposition 2.4], determines a lower bound on the reduced complexity of a nonempty subset  $S \in \mathcal{A}([\ell, r[)$ .

**Proposition 3.** *Let  $S \in \mathcal{A}([\ell, r[)$  be a subset composed of  $n$  disjoint semi-intervals. Then  $\Pi(S) > \frac{n}{4\sqrt{d}}$ .*

## 4.2 Return Times

Let  $T$  be an interval exchange transformation. For a subset  $S \in \mathcal{A}([\ell, r[)$  we define the maximal positive and maximal negative return times for  $T$  on  $S$  by the functions

$$\rho^+(S) = \min \left\{ n \geq 1 \mid T^n(S) \subset \bigcup_{i=0}^{n-1} T^i(S) \right\},$$

and

$$\rho^-(S) = \min \left\{ m \geq 1 \mid T^m(S) \subset \bigcup_{i=0}^{m-1} T^{-i}(S) \right\}.$$

We also define the minimal positive and the minimal negative return times as

$$\sigma^+(S) = \min \{ n \geq 1 \mid T^n(S) \cap S \neq \emptyset \},$$

and

$$\sigma^-(S) = \min \{ m \geq 1 \mid T^{-m}(S) \cap S \neq \emptyset \},$$

Note that, when  $S$  is a semi-interval, we have  $\rho^+(S) = \max_{z \in S} \rho_{S,T}^+(z)$  and  $\sigma^+(S) = \min_{z \in S} \rho_{S,T}^+(z)$ . Symmetrically  $\rho^-(S) = \max_{z \in S} \rho_{S,T}^-(z) + 1$  and  $\sigma^-(S) = \min_{z \in S} \rho_{S,T}^-(z) + 1$ .

If  $T$  is minimal, it is clear that  $[\ell, r[ = \bigcup_{i=0}^{\rho^+(S)-1} T^i(S) = \bigcup_{i=0}^{\rho^-(S)-1} T^{-i}(S)$ .

Let  $\zeta, \eta$  be two functions. We write  $\zeta = O(\eta)$  if there exists a constant  $C$  such that  $|\zeta| \leq C|\eta|$ . We write  $\zeta = \Theta(\eta)$  if one has both  $\zeta = O(\eta)$  and  $\eta = O(\zeta)$ . Note that  $\Theta$  is an equivalence relation.

Boshernitzan and Carroll give in [5] two upper bounds for  $\rho^+(S)$  and  $\sigma^+(S)$  for a subset  $S$  (Theorems 2.5 and 2.6 respectively) and a more precise estimation when the subset is a semi-interval (Theorem 2.8). Some slight modifications of the proofs can be made so that the results hold also for  $\rho^-$  and  $\sigma^-$ . We summarize these estimates in the following theorem.

**Theorem 8.** *For every  $S \in \mathcal{A}([\ell, r[)$  one has  $\rho^+(S), \rho^-(S) = O(\Psi(S))$  and  $\sigma^+(S), \sigma^-(S) = O\left(\frac{1}{|S|}\right)$ . Moreover, if  $T$  is minimal and  $J$  is a semi-interval, then  $\rho^+(J) = \Theta(\rho^-(J)) = \Theta(\sigma^+(J)) = \Theta(\sigma^-(J)) = \Theta\left(\frac{1}{|J|}\right)$ .*

An immediate corollary of Theorem 8 is the following (see also Corollary 2.9 of [5]).

**Corollary 2.** *Let  $T$  be minimal and assume that*

$$\{T^i(z) \mid -m+1 \leq i \leq n-1\} \cap J = \emptyset$$

*for some point  $z \in [\ell, r[$ , some semi-interval  $J \subset [\ell, r[$  and some integers  $m, n \geq$*

*1. Then  $|J| = O\left(\frac{1}{\max\{m, n\}}\right)$*

*Proof.* By the hypothesis,  $z \notin \bigcup_{i=0}^{n-1} T^{-i}(J)$  we have  $\rho^-(J) \geq n$ . Then, using Theorem 8, we obtain  $|J| = \Theta\left(\frac{1}{\rho^-(J)}\right) = O\left(\frac{1}{n}\right)$ . Symmetrically, since  $\rho^+(J) \geq m$ , one has  $|J| = O\left(\frac{1}{m}\right)$ . Then  $|J| = O\left(\min\left\{\frac{1}{m}, \frac{1}{n}\right\}\right) = O\left(\frac{1}{\max\{m, n\}}\right)$ .  $\square$

### 4.3 Reduced Complexity of Admissible Semi-Intervals

In order to demonstrate the main theorem (Theorem 6), we prove some preliminary results concerning the reduced complexity of admissible semi-intervals.

Let  $T$  be an  $s$ -interval exchange transformation. Recall that we denote by  $\text{Sep}(T) = \{\gamma_i \mid 0 \leq i \leq s-1\}$  the set of separation points. For every  $n \geq 0$  define  $\mathcal{D}_n(T) = \bigcup_{i=0}^{n-1} T^{-i}(\text{Sep}(T))$  with the convention  $\mathcal{D}_0 = \emptyset$ .

Since  $\text{Sep}(T^{-1}) = T(\text{Sep}(T))$ , one has  $\mathcal{D}_n(T^{-1}) = T^{n-1}(\mathcal{D}_n(T))$ .

Given two integers  $m, n \geq 1$ , we can define  $\mathcal{D}_{m,n} = \mathcal{D}_m(T) \cup \mathcal{D}_n(T^{-1})$ . An easy calculation shows that

$$\mathcal{D}_{m,n}(T) = \bigcup_{i=-m+1}^n T^i(\text{Sep}(T)).$$

Observe also that  $\mathcal{D}_{m,n}(T) = T^n(\mathcal{D}_{m+n}(T)) = T^{-m+1}(\mathcal{D}_{m+n}(T))$ .

Denote by  $\mathcal{V}_{m,n}(T)$  the family of semi-intervals whose endpoints are in  $\mathcal{D}_{m,n}(T)$ . Put  $\mathcal{V}(T) = \bigcup_{m,n \geq 0} \mathcal{V}_{m,n}(T)$ .

Every admissible semi-interval belong to  $\mathcal{V}(T)$ , while the converse is not true.

**Theorem 9.**  $\Pi(J) = \Theta(1)$  for every semi-interval  $J$  admissible for  $T$ .

*Proof.* Let  $m, n$  be the two minimal integers such that  $J = [t, w[ \in \mathcal{V}_{m,n}(T)$ . Then  $t, w \in \{T^m(\gamma_i) \mid 1 \leq i \leq s\} \cup \{T^{-n}(\gamma_i) \mid 1 \leq i \leq s\}$ . Suppose, for instance,  $t = T^M(\gamma)$ , with  $M = \max\{m, n\}$  and  $\gamma$  a separation point. The other cases – i.e.  $t = T^{-M}(\gamma)$ ,  $w = T^M(\gamma)$  or  $w = T^{-M}(\gamma)$  – are proved similary.

The only semi-interval in  $\mathcal{V}_{0,0}(T)$  is  $[\ell, r[$  and clearly in this case the theorem is verified. Suppose then that  $J \in \mathcal{V}_{m,n}(T)$  for some non-negative integers  $m, n$  with  $m+n > 0$ .

We have  $\Psi(J) = \max\{\Psi(t), \Psi(w)\} \leq Mu$  where  $u$  is the constant introduced in Proposition 2.

Moreover, by the definition of admissibility one has  $\{T^j(\gamma) \mid 1 \leq j \leq M\} \cap J = \emptyset$ . Thus, by Corollary 2 we have  $|J| = O\left(\frac{1}{M}\right)$ . Then  $\Pi(J) = |J| \Psi(J) = O(1)$ . By Proposition 3 we have  $\Pi(J) > \frac{1}{4\sqrt{d}}$ . This concludes the proof.  $\square$

Denote by  $\mathcal{U}_{m,n}(T)$  the family of semi-intervals partitioned by  $\mathcal{D}_{m,n}(T)$ .

Clearly  $\mathcal{V}_{m,n}(T)$  contains  $\mathcal{U}_{m,n}(T)$ . Indeed every semi-interval  $J \in \mathcal{V}_{m,n}(T)$  is a finite union of contiguous semi-intervals belonging to  $\mathcal{U}_{m,n}(T)$ .

Note that  $\mathcal{U}_{m,0}(T)$  is the family of semi-intervals exchanged by  $T^m$ , while  $\mathcal{U}_{0,n}(T)$  is the family of semi-intervals exchanged by  $T^{-n}$ .

Put  $\mathcal{U}(T) = \bigcup_{m,n \geq 0} \mathcal{U}_{m,n}(T)$ . Using Theorem 9 we easily deduce the following corollary, which is a generalization of Theorem 2.11 in [5].

**Corollary 3.**  $\Pi(J) = \Theta(1)$  for every semi-interval  $J \in \mathcal{U}(T)$ .

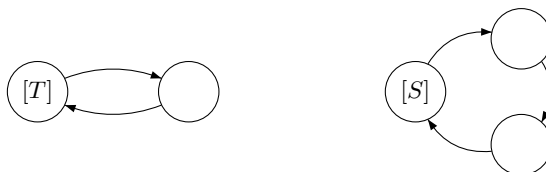
We are now able to prove Theorem 6.

*Proof of Theorem 6.* By Theorem 5, every admissible semi-interval can be obtained by a finite sequence  $\chi$  of right and left Rauzy inductions. Thus we can enumerate the family of all admissible semi-intervals. The conclusion follows easily from Theorem 7 and Theorem 9.  $\square$

An immediate corollary of Theorem 6 is the following.

**Corollary 4.** Let  $T$  be a regular interval exchange transformation defined over a quadratic field. Then the extension graph  $G(T)$  is finite.

*Example 9.* Let  $T = T_{\pi,(\beta,1-\beta)}$  and  $S = T_{\pi,(\gamma,1-\gamma)}$  be two regular 2-interval exchange transformations, where  $\pi = (12)$  is the permutation defined in Example 1,  $\beta = (2 - \sqrt{2})$  and  $\gamma = \frac{3-\sqrt{3}}{2}$ . The equivalence graphs of  $T$  and  $S$  are represented in Figure 5. Note that the ratio of the lengths of the semi-intervals exchanged by  $T$  is  $\frac{\beta}{1-\beta} = \sqrt{2} = 1 + \frac{1}{2+\frac{1}{2+\dots}} = 1 + \frac{1}{1+\sqrt{2}}$ , while the the ratio of the lengths of the semi-intervals exchanged by  $S$  is  $\frac{\gamma}{1-\gamma} = \sqrt{3} = 1 + \frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{2+\dots}}}} = 1 + \frac{1}{1+\sqrt{3}}$ .



**Fig. 5.** Equivalence graphs of the transformations  $T = T_{\pi,(\beta,1-\beta)}$  (on the left) and  $S = T_{\pi,(\gamma,1-\gamma)}$  (on the right).

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